

Isometry generators and canonical quantization on de Sitter spacetimes

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Abstract

The properties of the covariant fields on the de Sitter spacetimes are investigated focusing on the isometry generators and Casimir operators in order to establish the equivalence among the covariant representations and the unitary irreducible ones of the de Sitter isometry group. For the Dirac field it is shown that the spinor covariant representation transforming the Dirac field under de Sitter isometries is equivalent to a direct sum of two unitary irreducible representations of the $Sp(2, 2)$ group, transforming alike the particle and antiparticle field operators in momentum representation. Their basis generators and Casimir operators are written down finding that the covariant representations are equivalent to unitary irreducible ones from the principal series whose canonical labels are determined by the fermion mass and spin.

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Preface

The principal invariant quantities that determine the nature of the elementary particles are the mass and spin. In the quantum theory of free fields the mass and spin are determined exclusively by the invariants of the space-time symmetries in the sense that these are encapsulated in the eigenvalues of the Casimir operators of the unitary irreducible representations (UIRs) of the isometry group. Moreover, the interactions that are governed by other symmetries, the internal and gauge ones, cannot affect the meaning of these geometric invariants even though the mass has to be redefined in the renormalization procedure.

On the other hand, we know that the covariant free fields transform under isometries according to covariant representations (CRs) induced by the gauge group that leave invariant the metric of the flat model. This means that these CRs must be equivalent with some UIRs whose invariants have to define the mass and spin assuring thus the coherence of the entire theory.

In special relativity the Wigner theory of induced representations of the Poincaré group solves successfully this problem. However, what happens on curved spacetimes? How the mass and spin could be related to some invariants of spacetime symmetries? Obviously, we are far from a global solution of this problem such that we have to continue this investigation analyzing step by step different free fields in various geometries.

In this report we would like to point out that similar results as in the Wigner theory can be obtained at least for the Dirac field on the de Sitter spacetime.

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1 Introduction

In general relativity there are covariant quantum fields defined on a curved manifold (M, g) which transform under isometries according to CRs *induced* by finite-dimensional representations (reps.) of the universal covering group \hat{G} of the gauge one, G [1, 2]. In the case of four-dimensional local-Minkowskian manifolds under consideration here, these groups are $G = SO(1, 3)$ and respectively $\hat{G} = SL(2, \mathbb{C})$. For this reason, the spin terms of the operators generating CRs are given by linear reps. of the $sl(2, \mathbb{C})$ algebra instead of the $s(M)$ algebra of the universal covering group $S(M)$ of the isometry one, $I(M)$.

On the other hand, the group $S(M)$ has UIRs that may be related to the CRs. In special relativity, the CRs are equivalent to orthogonal sums of Wigner's UIRs that govern the transformation rules under isometries of the particle and antiparticle operators in momentum rep. [3, 4, 5]. This result is related to the special structure of the Poincaré isometry group, $T(4) \otimes G$ such that it cannot be generalized to other manifolds, not even in the case of the de Sitter spacetime which still allows a momentum rep..

The de Sitter manifold, denoted from now by M , is local-Minkowskian and has the isometry group $I(M) = SO(1, 4)$ which is the gauge group of the Minkowskian five-dimensional manifold M^5 embedding M . The UIRs of the corresponding group $S(M) = \text{Spin}(1, 4) = Sp(2, 2)$ are well-studied [6, 7] and used in various applications. Many authors exploited this high symmetry for building quantum theories, either by constructing symmetric two-point functions, avoiding thus the canonical quantization [8, 9], or by using directly these UIRs for finding field Eqs. without considering CRs [10, 11, 12]. Another approach which applies the *canonical quantization* to the covariant fields transforming according to *induced* CRs was initiated by Nachtmann [13] many years ago and continued in few of our papers [14, 15, 16, 17, 18].

In the present brief report we adopt this last framework for presenting our principal results concerning the role of the generators of the CRs and UIRs of the de Sitter isometries in determining the principal invariants of the quantum field theory (QFT) on the de Sitter spacetime [17, 19].

On this manifold we cannot apply the Wigner method for investigating the structure of the covariant fields such that we may study the general features of the CRs in configurations deriving the principal invariants. Thus we find the form of the generators of the CRs and the corresponding Casimir operators that help us to establish indirectly the equivalence of the induced CRs with orthogonal sums of UIRs of the

group $S(M)$ [17]. However, at this level, we cannot deduce the form of the UIRs generators in momentum rep. which act on the particle and antiparticle field operators. This is because of the absence of a general Wigner theory that forces us to use field equations for determining the structure of the covariant fields in each particular case separately.

For this reason we concentrated on the Dirac field investigating the relation among the CRs and UIRs of the Dirac theory in momentum rep. on the de Sitter background [19]. As mentioned, these CRs are induced by the linear reps. of the group \hat{G} without to meet explicitly the linear reps. of the group $S(M)$. Nevertheless, the equivalence between the Dirac CR and a pair of the UIRs of the group $S(M)$ can be proved constructing the UIR generators in momentum rep. as one-particle operators of the QFT and deriving the Casimir operators. In this manner we found that the Dirac particle and antiparticle operators in momentum rep. transform according to the *same* UIR that can be one of the equivalent UIRs (s, q) , from the principal series [6, 7], labeled by the spin $s = \frac{1}{2}$ and $q = \frac{1}{2} \pm i\frac{m}{\omega}$ where m is the fermion mass and ω is the Hubble constant of M in our notation. Note that the fundamental spinors we use here correspond to a fixed vacuum of the Bunch-Davies type [20] as in our de Sitter QED [18].

This result is similar to that of special relativity where the particle and antiparticle operators in momentum rep. of any covariant quantum field with unique spin, s , transform alike under Poincaré isometries, according to the same Wigner UIR induced by the $2s + 1$ -dimensional UIR of the group $SU(2)$ [21]. Obviously, this happens only if we respect the spin-statistic connection, assuming that the Dirac particle and antiparticle operators satisfy canonical anti-commutation rules.

On the other hand, the concrete forms of the generators presented here allowed us to expand in momentum rep. the conserved one-particle operators corresponding to the de Sitter isometries via Noether theorem. We showed that for all these operators (energy, momentum, angular momentum, etc.) the contributions of the particles and antiparticles are *additive* in contrast with the conserved charge where these have opposite signs. We find thus that an important feature of the QFT on Minkowski spacetime can be retrieved on curved backgrounds.

This paper is organized as follows. In the next section we present our general theory of covariant fields on curved spacetimes and the method of canonical quantization of a Lagrangian QFT in a given rep.. The CRs of special relativity and the Wigner theory of the induced UIRs of the Poincaré group are discussed in the third section. The next section is devoted to the theory of covariant fields on de Sitter spacetime, presenting

the general form of the generators of the CRs of the de Sitter isometries and the corresponding Casimir operators. In the fifth section we concentrate on the Dirac field on de Sitter spacetime giving the fundamental solutions of the free Dirac equation and deriving the generators of the UIRs in momentum rep. and the components of the Pauli-Lubanski operator which helps us to obtain the Casimir operators. Moreover, we give the general form of the momentum expansions of the principal conserved observables of QFT discussing briefly their properties. Finally, we present our conclusions.

2 Covariant quantum fields

Any local-Minkowskian spacetime (M, g) may be equipped with *local* frames $\{x; e\}$ formed by a local chart (or natural frame) $\{x\}$ and a non-holonomic orthogonal frame $\{e\}$. In a given local chart of coordinates x^μ , labelled by natural indices, $\mu, \nu, \dots = 0, 1, 2, 3$, the orthogonal frames and the corresponding coframes, $\{\hat{e}\}$, are defined by the tetrad fields $e_{\hat{\mu}}$ and $\hat{e}^{\hat{\mu}}$, which are labelled by local indices, $\hat{\mu}, \hat{\nu}, \dots = 0, 1, 2, 3$, and obey the usual duality, $\hat{e}^{\hat{\mu}}_\alpha e^\alpha_{\hat{\nu}} = \delta^{\hat{\mu}}_{\hat{\nu}}$, $\hat{e}^{\hat{\mu}}_\alpha e^\beta_{\hat{\mu}} = \delta^\beta_\alpha$, and orthonormalization, $e_{\hat{\mu}} \cdot e_{\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$, $\hat{e}^{\hat{\mu}} \cdot \hat{e}^{\hat{\nu}} = \eta^{\hat{\mu}\hat{\nu}}$, conditions.

The tetrad fields define the local derivatives $\hat{\partial}_{\hat{\alpha}} = e^\mu_{\hat{\alpha}} \partial_\mu$ and the basis 1-forms $\tilde{\omega}^{\hat{\alpha}}(x) = \hat{e}^{\hat{\alpha}}_\mu(x) dx^\mu$ giving the metric tensor $g_{\mu\nu} = \eta_{\hat{\alpha}\hat{\beta}} \hat{e}^{\hat{\alpha}}_\mu \hat{e}^{\hat{\beta}}_\nu$ that raises or lowers the natural indices while for the local ones we have to use the flat metric $\eta = \text{diag}(1, -1, -1, -1)$ of the Minkowski spacetime (M_0, η) which is the pseudo-Euclidean model of (M, g) .

2.1 Covariant fields on curved spacetimes

The metric η remains invariant under the transformations of the group $O(1, 3)$ which includes the Lorentz group, L_+^\uparrow , whose universal covering group is $SL(2, \mathbb{C})$. In the usual covariant parametrization, with the real parameters, $\omega^{\hat{\alpha}\hat{\beta}} = -\omega^{\hat{\beta}\hat{\alpha}}$, the transformations

$$A(\omega) = \exp\left(-\frac{i}{2}\omega^{\hat{\alpha}\hat{\beta}}S_{\hat{\alpha}\hat{\beta}}\right) \in SL(2, \mathbb{C}) \quad (1)$$

depend on the covariant basis-generators of the $sl(2, \mathbb{C})$ Lie algebra, $S_{\hat{\alpha}\hat{\beta}}$, which are the principal spin operators generating all the spin terms of other operators. In this parametrization the matrix elements in local frames of the transformations $\Lambda(\omega) \equiv \Lambda[A(\omega)] \in L_+^\uparrow$ associated to $A(\omega)$

through the canonical homomorphism can be expanded as $\Lambda^{\hat{\mu}\cdot}_{\cdot\hat{\nu}}(\omega) = \delta^{\hat{\mu}}_{\hat{\nu}} + \omega^{\hat{\mu}\cdot}_{\cdot\hat{\nu}} + \dots$. Obviously, $\Lambda(0) = I$ is the identity transformation of L_+^\dagger .

Assuming that (M, g) is orientable and time-orientable we can restrict ourselves to consider $G(\eta) = L_+^\dagger$ as the *gauge group* of the Minkowski metric η . In this framework one can build the gauge-covariant (or, simply, covariant) field theories whose physical meaning does not depend on the local frames one uses.

The *covariant fields*, $\psi_{(\rho)} : M \rightarrow \mathcal{V}_{(\rho)}$, are locally defined over M with values in the vector spaces $\mathcal{V}_{(\rho)}$ carrying the finite-dimensional non-unitary reps. ρ of the group $SL(2, \mathbb{C})$. In general, these reps. are reducible being equivalent to direct sums of irreducible ones, (j_1, j_2) [5]. They determine the form of the covariant derivatives of the field $\psi_{(\rho)}$ in local frames,

$$D_{\hat{\alpha}}^{(\rho)} = e_{\hat{\alpha}}^\mu D_\mu^{(\rho)} = \hat{\partial}_{\hat{\alpha}} + \frac{i}{2} \rho(S^{\hat{\beta}\cdot}_{\cdot\hat{\gamma}}) \hat{\Gamma}_{\hat{\alpha}\hat{\beta}}^{\hat{\gamma}}. \quad (2)$$

The local connection coefficients $\hat{\Gamma}_{\hat{\mu}\hat{\nu}}^{\hat{\sigma}} = e_{\hat{\mu}}^\alpha e_{\hat{\nu}}^\beta (\hat{e}_\gamma^{\hat{\sigma}} \Gamma_{\alpha\beta}^\gamma - \hat{e}_{\beta,\alpha}^{\hat{\sigma}})$ assure the covariance of the whole theory under the (point-dependent) tetrad-gauge transformations,

$$\tilde{\omega} \rightarrow \tilde{\omega}' = \Lambda[A] \tilde{\omega}, \quad (3)$$

$$\psi_{(\rho)} \rightarrow \psi'_{(\rho)} = \rho(A) \psi_{(\rho)}, \quad (4)$$

produced by the automorphisms $A \in SL(2, \mathbb{C})$ of the spin fiber bundle.

When (M, g) has isometries, $x \rightarrow x' = \phi_{\mathfrak{g}}(x)$, given by the (non-linear) rep. $\mathfrak{g} \rightarrow \phi_{\mathfrak{g}}$ of the isometry group $I(M)$ defined by the composition rule $\phi_{\mathfrak{g}} \circ \phi_{\mathfrak{g}'} = \phi_{\mathfrak{g}\mathfrak{g}'}$, $\forall \mathfrak{g}, \mathfrak{g}' \in I(M)$. Then we denote by $id = \phi_{\mathfrak{e}}$ the identity function, corresponding to the unit $\mathfrak{e} \in I(M)$, and deduce $\phi_{\mathfrak{g}}^{-1} = \phi_{\mathfrak{g}^{-1}}$. In a given parametrization, $\mathfrak{g} = \mathfrak{g}(\xi)$ (with $\mathfrak{e} = \mathfrak{g}(0)$), the isometries

$$x \rightarrow x' = \phi_{\mathfrak{g}(\xi)}(x) = x + \xi^a k_a(x) + \dots \quad (5)$$

lay out the Killing vectors $k_a = \partial_{\xi^a} \phi_{\mathfrak{g}(\xi)}|_{\xi=0}$ associated to the parameters ξ^a ($a, b, \dots = 1, 2, \dots, N$).

In general, the isometries may change the relative position of the local frames affecting thus the physical interpretation. For this reason we proposed the theory of external symmetry [1] where we introduced the combined transformations $(A_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ able to correct the position of the local frames. These transformations must preserve not only the metric but the tetrad-gauge too, transforming the 1-forms as $\tilde{\omega}(x') = \Lambda[A_{\mathfrak{g}}(x)] \tilde{\omega}(x)$. Hereby, we deduce [1],

$$\Lambda^{\hat{\alpha}\cdot}_{\cdot\hat{\beta}}[A_{\mathfrak{g}}(x)] = \hat{e}_{\hat{\mu}}^{\hat{\alpha}}[\phi_{\mathfrak{g}}(x)] \frac{\partial \phi_{\mathfrak{g}}^\mu(x)}{\partial x^\nu} e_{\hat{\beta}}^\nu(x), \quad (6)$$

assuming, in addition, that $A_{\mathfrak{g}=\epsilon}(x) = 1 \in SL(2, \mathbb{C})$. We obtain thus the desired transformation laws,

$$(A_{\mathfrak{g}}, \phi_{\mathfrak{g}}) : \quad \begin{aligned} e(x) &\rightarrow e'(x') = e[\phi_{\mathfrak{g}}(x)], \\ \psi_{(\rho)}(x) &\rightarrow \psi'_{(\rho)}(x') = \rho[A_{\mathfrak{g}}(x)]\psi_{(\rho)}(x). \end{aligned} \quad (7)$$

that preserve the tetrad-gauge. We have shown that the pairs $(A_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ constitute a well-defined Lie group that can be seen as a rep. of the universal covering group of $I(M)$ denoted here by $S(M)$ [1].

In a given parametrization, $\mathfrak{g}(\xi)$, for small values of ξ^a , the $SL(2, \mathbb{C})$ parameters of $A_{\mathfrak{g}(\xi)}(x) \equiv A[\omega_{\xi}(x)]$ can be expanded as $\omega_{\xi}^{\hat{\alpha}\hat{\beta}}(x) = \xi^a \Omega_a^{\hat{\alpha}\hat{\beta}}(x) + \dots$, in terms of the functions

$$\Omega_a^{\hat{\alpha}\hat{\beta}} \equiv \frac{\partial \omega_{\xi}^{\hat{\alpha}\hat{\beta}}}{\partial \xi^a} \Big|_{\xi=0} = \left(\hat{e}_{\mu}^{\hat{\alpha}} k_{a,\nu}^{\mu} + \hat{e}_{\nu,\mu}^{\hat{\alpha}} k_a^{\mu} \right) e_{\lambda}^{\nu} \eta^{\lambda\hat{\beta}} \quad (8)$$

which are skew-symmetric, $\Omega_a^{\hat{\alpha}\hat{\beta}} = -\Omega_a^{\hat{\beta}\hat{\alpha}}$, only when k_a are Killing vectors [1].

The last of Eqs. (7) defines the CRs induced by the finite-dimensional rep. ρ of the group $SL(2, \mathbb{C})$. These are operator-valued reps., $T^{(\rho)} : (A_{\mathfrak{g}}, \phi_{\mathfrak{g}}) \rightarrow T_{\mathfrak{g}}^{(\rho)}$, of the group $S(M)$ whose covariant transformations,

$$(T_{\mathfrak{g}}^{(\rho)} \psi_{(\rho)})[\phi_{\mathfrak{g}}(x)] = \rho[A_{\mathfrak{g}}(x)]\psi_{(\rho)}(x), \quad (9)$$

leave the field Eq. invariant since their basis-generators [1],

$$X_a^{(\rho)} = i\partial_{\xi^a} T_{\mathfrak{g}(\xi)}^{(\rho)} \Big|_{\xi=0} = -ik_a^{\mu} \partial_{\mu} + \frac{1}{2} \Omega_a^{\hat{\alpha}\hat{\beta}} \rho(S_{\hat{\alpha}\hat{\beta}}), \quad (10)$$

commute with the operator of the field Eq.. Moreover, these generators satisfy the commutation rules $[X_a^{(\rho)}, X_b^{(\rho)}] = ic_{abc} X_c^{(\rho)}$ determined by the structure constants, c_{abc} , of the algebras $s(M) \sim i(M)$. In other words, they are the basis-generators of a CR of the $s(M)$ algebra *induced* by the rep. ρ of the $sl(2, \mathbb{C})$ algebra. These generators can be put in (general relativistic) covariant form either in non-holonomic frames [1] or even in holonomic ones [2], generalizing thus the formula given by Carter and McLenaghan for the Dirac field [22].

The generators (10) have, in general, point-dependent spin terms which do not commute with the orbital parts. However, there are tetrad-gauges in which at least the generators of a subgroup $H \subset I(M)$ may have point-independent spin terms commuting with the orbital parts. Then we say that the restriction to H of the CR $T^{(\rho)}$ is *manifest* covariant [1]. Obviously, if $H = I(M)$ then the whole rep. $T^{(\rho)}$ is manifest covariant. In particular, the linear CRs on the Minkowski spacetime have this property.

2.2 Lagrangian formalism and quantization

The construction of the Lagrangian theory requires to use positive defined quantities, invariant under the action of the transformations $\rho(A)$. Since the finite-dimensional reps. ρ of the $SL(2, \mathbb{C})$ group are non-unitary we must introduce the (generalized) Dirac conjugation, $\bar{\psi}_{(\rho)} = \psi_{(\rho)}^+ \gamma_{(\rho)}$, where the matrix $\gamma_{(\rho)} = \gamma_{(\rho)}^+ = \gamma_{(\rho)}^{-1}$ satisfies $\bar{\rho}(A) = \gamma_{(\rho)} \rho(A)^+ \gamma_{(\rho)} = \rho(A^{-1})$. Then the form $\bar{\psi}_{(\rho)} \psi_{(\rho)}$ is invariant under the gauge transformations (4).

In general, the Dirac conjugation can be defined for the reducible reps. ρ which are direct sums including only pairs of irreducible adjoint reps., $\rho = \dots(j_1, j_2) \oplus (j_2, j_1) \dots$ as we briefly argue in Appendix A. For example, the covariant fields with *unique* spin s are constructed in the vector spaces of the reps. $\rho(s) = (s, 0) \oplus (0, s)$ [21].

The covariant free fields satisfy field equations which can be derived from actions of the form

$$\mathcal{S}[\psi_{(\rho)}, \bar{\psi}_{(\rho)}] = \int_{\Delta} d^4x \sqrt{g} \mathcal{L}(\psi_{(\rho)}, \psi_{(\rho);\mu}, \bar{\psi}_{(\rho)}, \bar{\psi}_{(\rho);\mu}), \quad g = |\det g_{\mu\nu}|, \quad (11)$$

depending on the field $\psi_{(\rho)}$, its Dirac adjoint $\bar{\psi}_{(\rho)}$ and their corresponding covariant derivatives $\psi_{(\rho);\mu} = D_{\mu} \psi_{(\rho)}$ and $\bar{\psi}_{(\rho);\mu} = \overline{D_{\mu} \psi_{(\rho)}}$ defined by the rep. ρ of the group $SL(2, \mathbb{C})$. The action \mathcal{S} is extremal if the covariant fields satisfy the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}_{(\rho)}} - \frac{1}{\sqrt{g}} \partial_{\mu} \frac{\partial(\sqrt{g} \mathcal{L})}{\partial \bar{\psi}_{(\rho);\mu}} = 0, \quad \frac{\partial \mathcal{L}}{\partial \psi_{(\rho)}} - \frac{1}{\sqrt{g}} \partial_{\mu} \frac{\partial(\sqrt{g} \mathcal{L})}{\partial \psi_{(\rho);\mu}} = 0. \quad (12)$$

Any transformation $\psi_{(\rho)} \rightarrow \psi'_{(\rho)} = \psi_{(\rho)} + \delta \psi_{(\rho)}$ leaving the action invariant, $\mathcal{S}[\psi'_{(\rho)}, \bar{\psi}'_{(\rho)}] = \mathcal{S}[\psi_{(\rho)}, \bar{\psi}_{(\rho)}]$, is a symmetry transformation. The Noether theorem shows that each symmetry transformation gives rise to the current

$$\Theta^{\mu} \propto \delta \bar{\psi}_{(\rho)} \frac{\partial \mathcal{L}}{\partial \bar{\psi}_{(\rho);\mu}} + \frac{\partial \mathcal{L}}{\partial \psi_{(\rho);\mu}} \delta \psi_{(\rho)} \quad (13)$$

which is conserved in the sense that $\Theta^{\mu}_{;\mu} = 0$.

In the case of isometries, transforming simultaneously the coordinates and the field components according to Eq. (7), we must take $\delta \psi_{(\rho)} = -i \xi^a X_a^{(\rho)} \psi_{(\rho)}$ where $X_a^{(\rho)}$ are defined by Eq. (10). Consequently, each isometry of parameter ξ^a give rise to the corresponding conserved current

$$\Theta_a^{\mu} = i \left(\overline{X_a^{(\rho)} \psi_{(\rho)}} \frac{\partial \mathcal{L}}{\partial \bar{\psi}_{(\rho);\mu}} - \frac{\partial \mathcal{L}}{\partial \psi_{(\rho);\mu}} X_a^{(\rho)} \psi_{(\rho)} \right), \quad a = 1, 2 \dots N. \quad (14)$$

Then we may define the relativistic scalar product $\langle \cdot, \cdot \rangle$ as

$$\langle \psi, \psi' \rangle = i \int_{\partial\Delta} d\sigma_\mu \sqrt{g} \left(\bar{\psi} \frac{\partial \mathcal{L}'}{\partial \bar{\psi}'_{,\mu}} - \frac{\partial \mathcal{L}}{\partial \psi_{,\mu}} \psi' \right), \quad (15)$$

such that the conserved quantities (charges) can be represented as expectation values of the symmetry generators,

$$C_a = \int_{\partial\Delta} d\sigma_\mu \sqrt{g} \Theta_a^\mu = \langle \psi_{(\rho)}, X_a^{(\rho)} \psi_{(\rho)} \rangle, \quad (16)$$

Notice that the operators X are self-adjoint with respect to this scalar product, i. e. $\langle X\psi, \psi' \rangle = \langle \psi, X\psi' \rangle$.

The operator algebra freely generated by the isometry generators offers us the conserved operators that commutes with the operator of the field Eq. \mathcal{E} . Other conserved operators are the generators of the internal symmetries we ignore here since we study how the particles are defined by the external symmetry. Thus we restrict ourselves to the whole algebra of the isometry generators from which we may select the sets of commuting operators $\{A_1, A_2, \dots, A_n\}$ determining the fundamental solutions of particles, $U_\alpha \in \mathcal{F}^+$, and antiparticles, $V_\alpha \in \mathcal{F}^-$, that satisfy the field Eq. and, in addition, depend on the set of the corresponding eigenvalues $\alpha = \{a_1, a_2, \dots, a_n\}$ spanning a discrete or continuous spectra of the common eigenvalue problems

$$A_i U_\alpha = a_i U_\alpha, \quad A_i V_\alpha = -a_i V_\alpha, \quad i = 1, 2, \dots, n. \quad (17)$$

We stress that these rules allow us to separate the subspace of particle fundamental solutions \mathcal{F}^+ from that of the antiparticle ones \mathcal{F}^- . However, this separation is unique only when the commuting operators form a *complete* system. Otherwise, there are many possibilities of separation each one defining its own vacuum state as it happens, for example, in the de Sitter case. Then supplemental criteria will help us to fix the vacuum state.

After determining the fundamental solutions we may write the mode expansion of the covariant field as

$$\psi_{(\rho)}(x) = \int_{\alpha \in \Sigma} U_\alpha(x) a(\alpha) + V_\alpha(x) b^*(\alpha), \quad (18)$$

where we sum over the discrete part (Σ_d) and integrate over the continuous part (Σ_c) of the spectrum $\Sigma = \Sigma_d \cup \Sigma_c$.

The fundamental solutions are orthogonal with respect to the relativistic scalar product and can be normalized such that

$$\langle U_\alpha, U_{\alpha'} \rangle = \pm \langle V_\alpha, V_{\alpha'} \rangle = \delta(\alpha, \alpha') = \begin{cases} \delta_{\alpha, \alpha'}, & \alpha, \alpha' \in \Sigma_d \\ \delta(\alpha - \alpha'), & \alpha, \alpha' \in \Sigma_c \end{cases} \quad (19)$$

$$\langle U_\alpha, V_{\alpha'} \rangle = \langle V_\alpha, U_{\alpha'} \rangle = 0. \quad (20)$$

Note that the sign $+$ arises for fermions while the sign $-$ is obtained for bosons. Under such circumstances, the conserved quantities get a correct physical meaning only if we perform the second quantization postulating that a and b become field operators which satisfy the canonical non-vanishing rules

$$[a(\alpha), a^\dagger(\alpha')]_\pm = [b(\alpha), b^\dagger(\alpha')]_\pm = \delta(\alpha, \alpha'), \quad (21)$$

where we denote $[x, y]_\pm = xy \pm yx$. Then the fields $\psi_{(\rho)}$ become quantum fields (with b^\dagger instead of b^*) while the conserved quantities (16) become one-particle operators,

$$C_a \rightarrow \mathbf{X}_a^{(\rho)} =: \langle \psi_{(\rho)}, X_a^{(\rho)} \psi_{(\rho)} \rangle : \quad (22)$$

calculated respecting the normal ordering of the operator products [23]. Thanks to this method of quantization these operators are now the isometry generators of the quantum field theory. More specific, the one particle operators $\mathbf{X}_a^{(\rho)}$ are the basis generators of a rep. of the algebra $s(M)$ with values in operator algebra.

In a similar manner one can define the generators of the internal symmetries as for example the charge one-particle operator $\mathbf{Q} =: \langle \psi, \psi \rangle :$. Thus we obtain a reach operator algebra formed by field operators and one-particle ones which have the obvious properties

$$[\mathbf{X}, \psi(x)] = -(X\psi)(x), \quad [\mathbf{X}, \mathbf{Y}] =: \langle \psi, ([X, Y]\psi) \rangle : . \quad (23)$$

In general, if the one-particle operator X does not mix among themselves the subspaces of fundamental solutions \mathcal{F}^+ and \mathcal{F}^- then it can be expanded as

$$\begin{aligned} \mathbf{X} &= : \langle \psi, X\psi \rangle := \mathbf{X}^{(+)} + \mathbf{X}^{(-)} \\ &= \int_{\alpha \in \Sigma} \int_{\alpha' \in \Sigma} \tilde{X}^{(+)}(\alpha, \alpha') a^\dagger(\alpha) a(\alpha') + \tilde{X}^{(-)}(\alpha, \alpha') b^\dagger(\alpha) b(\alpha'), \end{aligned} \quad (24)$$

where

$$\tilde{X}^{(+)}(\alpha, \alpha') = \langle U_\alpha, XU_{\alpha'} \rangle, \quad \tilde{X}^{(-)}(\alpha, \alpha') = \langle V_\alpha, XV_{\alpha'} \rangle. \quad (25)$$

When there are differential operators $\tilde{X}^{(\pm)}$ acting on the continuous variables of the set $\{\alpha\}$ such that $\tilde{X}^{(\pm)}(\alpha, \alpha') = \delta(\alpha, \alpha')\tilde{X}^{(\pm)}$ we say that $\tilde{X}^{(\pm)}$ are the operators of the rep. $\{\alpha\}$ (in the sense of the relativistic QM).

We stress that all the isometry generators have this property such that the corresponding operators $\tilde{X}_a^{(\pm)}$ are the basis generators of the isometry transformations of the field operators a and b . However, the algebraic relations (21) remain invariant only if a and b transform according to UIRs of the isometry group. A crucial problem is now the relation between the CR transforming the covariant field $\psi_{(\rho)}$ and the set of UIRs transforming the particle and antiparticle operators a and b . This problem, referred here as the CR-UIR equivalence, is successfully solved in special relativity thanks to the Wigner theory of induced reps. of the Poincaré group.

3 Covariant fields in special relativity

On the Minkowski flat spacetime, (M_0, η) , the fields $\psi_{(\rho)}$ transform under isometries according to *manifest* covariant reps. in *inertial* (local) frames defined by $e_\nu^\mu = \hat{e}_\nu^\mu = \delta_\nu^\mu$. The isometries are just the transformations $x \rightarrow x' = \Lambda[A(\omega)]x - a$ of the Poincaré group $I(M_0) = \mathcal{P}_+^\uparrow = T(4) \mathbf{s} L_+^\uparrow$ [21] whose universal covering group is $S(M_0) = \tilde{\mathcal{P}}_+^\uparrow = T(4) \mathbf{s} SL(2, \mathbb{C})$. Both these groups are semidirect products (denoted by \mathbf{s}) where the translations form the *normal* Abelian subgroup $T(4)$.

3.1 Generators of manifest CRs

The manifest CRs, $T^{(\rho)} : (A, a) \rightarrow T_{A,a}^{(\rho)}$, of the group $S(M_0)$ have the transformation rules

$$(T_{A,a}^{(\rho)}\psi_{(\rho)})(x) = \rho(A)\psi_{(\rho)}(\Lambda(A)^{-1}(x+a)), \quad (26)$$

and the well-known basis-generators of the $s(M_0)$ algebra,

$$\hat{P}_\mu \equiv \hat{X}_{(\mu)}^{(\rho)} = i\partial_\mu, \quad (27)$$

$$\hat{J}_{\mu\nu}^{(\rho)} \equiv \hat{X}_{(\mu\nu)}^{(\rho)} = i(\eta_{\mu\alpha}x^\alpha\partial_\nu - \eta_{\nu\alpha}x^\alpha\partial_\mu) + S_{\mu\nu}^{(\rho)}, \quad (28)$$

which have point-independent spin parts denoted by $S_{\hat{\mu}\hat{\nu}}^{(\rho)}$ instead of $\rho(S_{\hat{\mu}\hat{\nu}})$. Here it is convenient to separate the energy operator, $\hat{H} = \hat{P}_0$, and to write the $sl(2, \mathbb{C})$ generators as

$$\hat{J}_i^{(\rho)} = \frac{1}{2}\varepsilon_{ijk}\hat{J}_{jk}^{(\rho)} = -i\varepsilon_{ijk}x^j\partial_k + S_i^{(\rho)}, \quad S_i^{(\rho)} = \frac{1}{2}\varepsilon_{ijk}S_{jk}^{(\rho)}, \quad (29)$$

$$\hat{K}_i^{(\rho)} = \hat{J}_{0i}^{(\rho)} = i(x^i\partial_t + t\partial_i) + S_{0i}^{(\rho)}, \quad i, j, k \dots = 1, 2, 3, \quad (30)$$

denoting $\vec{S}^2 = S_i S_i$ and $\vec{S}_0^2 = S_{0i} S_{0i}$. Thus we lay out the standard basis of the $s(M_0)$ algebra, $\{\hat{H}, \hat{P}_i, \hat{J}_i^{(\rho)}, \hat{K}_i^{(\rho)}\}$.

The invariants of the manifest covariant fields are the eigenvalues of the Casimir operators of the reps. $T^{(\rho)}$ that read

$$\hat{C}_1 = \hat{P}_\mu \hat{P}^\mu, \quad \hat{C}_2^{(\rho)} = -\eta_{\mu\nu} \hat{W}^{(\rho)\mu} \hat{W}^{(\rho)\nu}, \quad (31)$$

where the Pauli-Lubanski operator [21],

$$\hat{W}^{(\rho)\mu} = -\frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} \hat{P}_\nu \hat{J}_{\alpha\beta}^{(\rho)}, \quad (32)$$

has the components

$$\hat{W}_0^{(\rho)} = \hat{J}_i^{(\rho)} \hat{P}_i = S_i^{(\rho)} \hat{P}_i, \quad \hat{W}_i^{(\rho)} = \hat{H} \hat{J}_i^{(\rho)} + \varepsilon_{ijk} \hat{K}_j^{(\rho)} \hat{P}_k, \quad (33)$$

resulting from Eqs. (27) and (28) where we take $\varepsilon^{0123} = -\varepsilon_{0123} = -1$.

In the Poincaré algebra we find the complete system of commuting operators $\{\hat{H}, \hat{P}_1, \hat{P}_2, \hat{P}_3\}$ defining the momentum rep.. The fundamental solutions are common eigenfunctions of this system such that any covariant quantum field can be written as

$$\psi_{(\rho)}(x) = \int d^3p \sum_{s\sigma} \left[U_{\vec{p},s\sigma}(x) a_{s\sigma}(\vec{p}) + V_{\vec{p},s\sigma}(x) b_{s\sigma}^\dagger(\vec{p}) \right] \quad (34)$$

where $a_{s\sigma}$ and $b_{s\sigma}$ are the field operators of a particle and antiparticle of *spin* s and *polarization* σ while the fundamental solutions have the form

$$U_{\vec{p},s\sigma}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} u_{s\sigma}(\vec{p}) e^{-iEt + i\vec{p}\cdot\vec{x}}, \quad V_{\vec{p},s\sigma}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} v_{s\sigma}(\vec{p}) e^{iEt - i\vec{p}\cdot\vec{x}}. \quad (35)$$

The vectors $u_{s\sigma}(\vec{p})$ and $v_{s\sigma}(\vec{p})$ have to be determined by the concrete form of the field Eq. and relativistic scalar product. However, when the field Eqs. are linear we can postulate the orthonormalization relations

$$\bar{u}_{s\sigma}(\vec{p}) u_{s'\sigma'}(\vec{p}) = \bar{v}_{s\sigma}(\vec{p}) v_{s'\sigma'}(\vec{p}) = \delta_{ss'} \delta_{\sigma\sigma'}, \quad (36)$$

$$\bar{u}_{s\sigma}(\vec{p}) v_{s'\sigma'}(\vec{p}) = \bar{v}_{s\sigma}(\vec{p}) u_{s'\sigma'}(\vec{p}) = 0. \quad (37)$$

that guarantee the separation of the particle and antiparticle sectors.

The first invariant (31a) gives the mass condition, $\hat{P}^2 \psi_{(\rho)} = m^2 \psi_{(\rho)}$, fixing the orbit in the momentum spaces on which the fundamental solutions are defined. For the massive fields of mass m the momentum spans the orbit $\Omega_m = \{\vec{p} | p^2 = m^2\}$ which means that $p_0 = \pm E$ where $E = \sqrt{m^2 + \vec{p}^2}$. The solutions U are considered of positive frequencies

having $p_0 = E$ while for the negative frequency ones, V , we must take $p_0 = -E$. In this manner the general rule (17) of separating the particle and antiparticle modes becomes

$$\hat{H}U_{\vec{p},s\sigma} = EU_{\vec{p},s\sigma}, \quad \hat{H}V_{\vec{p},s\sigma} = -EV_{\vec{p},s\sigma}, \quad (38)$$

$$\hat{P}^i U_{\vec{p},s\sigma} = p^i U_{\vec{p},s\sigma}, \quad \hat{P}^i V_{\vec{p},s\sigma} = -p^i V_{\vec{p},s\sigma}. \quad (39)$$

The second invariant is less relevant for the CRs since its form in configurations is quite complicated

$$\begin{aligned} \hat{\mathcal{C}}_2^{(\rho)} = & -(\vec{S}^{(\rho)})^2 \partial_t^2 + 2(iS_{0k}^{(\rho)} - \varepsilon_{ijk} S_i^{(\rho)} S_{0j}^{(\rho)}) \partial_k \partial_t \\ & - \left[(\vec{S}_0^{(\rho)})^2 \Delta - (S_i^{(\rho)} S_j^{(\rho)} + S_{0i}^{(\rho)} S_{0j}^{(\rho)}) \partial_i \partial_j \right]. \end{aligned} \quad (40)$$

Consequently, we may study its action in momentum rep. where it selects the induced Wigner UIRs equivalent with the manifest CR.

3.2 Wigner's induced UIRs

The Wigner theory of the induced UIRs is based on the fact that the orbits in momentum space may be built by using Lorentz transformations [3, 4]. In the case of massive particles we discuss here, any $\vec{p} \in \Omega_m$ can be obtained applying a suitable *boost* transformation $L_{\vec{p}} \in L_+^\uparrow$ to the *representative* momentum $\vec{p} = (m, 0, 0, 0)$ such that $\vec{p} = L_{\vec{p}} \vec{p}$. The rotations that leave \vec{p} invariant, $R\vec{p} = \vec{p}$, form the *stable* group $SO(3) \subset L_+^\uparrow$ whose universal covering group $SU(2)$ is called the *little* group associated to the representative momentum \vec{p} .

We observe that the boosts $L_{\vec{p}}$ are defined up to a rotation since $L_{\vec{p}} R \vec{p} = L_{\vec{p}} \vec{p}$. Therefore, these span the homogeneous space $L_+^\uparrow / SO(3)$. The corresponding transformations of the $SL(2, \mathbb{C})$ group are denoted by $A_{\vec{p}} \in SL(2, \mathbb{C}) / SU(2)$ assuming that these satisfy $\Lambda(A_{\vec{p}}) = L_{\vec{p}}$ and $A_{\vec{p}} = 1 \in SL(2, \mathbb{C})$.

In applications one prefers to chose genuine Lorentz transformations $A_{\vec{p}} = e^{-i\alpha n^i S_{0i}}$ with $\alpha = \text{arctanh} \frac{p}{E}$ and $n^i = \frac{p^i}{p}$ with $p = |\vec{p}|$. In the spinor rep. $\rho_s = (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ of the Dirac theory one finds [24]

$$\rho_s(A_{\vec{p}}) = \frac{E + m + \gamma^0 \gamma^i p^i}{\sqrt{2m(E + m)}}. \quad (41)$$

where γ^μ denote the Dirac matrices. The corresponding transformations of the L_+^\uparrow group, $L_{\vec{p}} = \Lambda(A_{\vec{p}})$, have the matrix elements

$$(L_{\vec{p}})^{0\cdot}_0 = \frac{E}{m}, \quad (L_{\vec{p}})^{0\cdot}_i = (L_{\vec{p}})^{i\cdot}_0 = \frac{p^i}{m}, \quad (L_{\vec{p}})^{i\cdot}_j = \delta_{ij} + \frac{p^i p^j}{m(E + m)}. \quad (42)$$

Furthermore, we look for the transformations in momentum rep. generated by the transformation rule (26) of the manifest CR under consideration. After a little calculation we obtain

$$\sum_{s'\sigma'} u_{s'\sigma'}(\vec{p})(T_{A,a}a_{s'\sigma'})(\vec{p}) = \sum_{s\sigma} \rho(A) u_{s\sigma}(\vec{p}') a_{s\sigma}(\vec{p}') e^{-ia\cdot p} \quad (43)$$

$$\sum_{s'\sigma'} v_{s'\sigma'}(\vec{p})(T_{A,a}b_{s'\sigma'}^\dagger)(\vec{p}) = \sum_{s\sigma} \rho(A) v_{s\sigma}(\vec{p}') b_{s\sigma}^\dagger(\vec{p}') e^{ia\cdot p} \quad (44)$$

where $a\cdot p = Ea^0 - \vec{p}\cdot\vec{a}$ and $\vec{p}' = \Lambda(A)^{-1}\vec{p}$. Focusing on the first Eq., we introduce the Wigner mode functions $u_{s\sigma}(\vec{p}) = \rho(A_{\vec{p}})\hat{u}_{s\sigma}$ where the vectors $\hat{u}_{s\sigma} \in \mathcal{V}_{(\rho)}$ are independent on \vec{p} and satisfy $\bar{u}_{s\sigma}\hat{u}_{s'\sigma'} = \bar{u}_{s\sigma}(\vec{p})u_{s'\sigma'}(\vec{p}) = \delta_{ss'}\delta_{\sigma\sigma'}$ according to Eq. (36). We obtain thus the transformation rule of the Wigner reps. *induced* by the subgroup $T(4)\mathbf{s}SU(2)$, that read [3, 5]

$$(T_{A,a}a_{s\sigma})(\vec{p}) = \sum_{\sigma'} D_{\sigma\sigma'}^s(A, \vec{p}) a_{s\sigma'}(\vec{p}') e^{i\vec{a}\cdot\vec{p}} \quad (45)$$

where

$$D_{\sigma\sigma'}^s(A, \vec{p}) = \bar{u}_{s\sigma}\rho[W(A, \vec{p})]\hat{u}_{s\sigma'}, \quad W(A, \vec{p}) = A_{\vec{p}}^{-1}AA_{\vec{p}'} \quad (46)$$

The Wigner transformations $W(A, \vec{p}) = A_{\vec{p}}^{-1}AA_{\vec{p}'}$ are of the little group $SU(2)$ since one can verify that $\Lambda[W(A, \vec{p})] = L_{\vec{p}}^{-1}\Lambda(A)L_{\vec{p}'} \in SO(3)$ leaving invariant the representative momentum \vec{p} . Therefore, the matrices D^s realize the UIR of spin (s) of the little group $SU(2)$ that induces the Wigner UIR (45) denoted by $(\pm m, s)$ [5]. Note that the role of the vectors $\hat{u}_{s\sigma}$ is to select the spin content of the CR determining the Wigner UIRs whose direct sum is equivalent to the manifest CR $T^{(\rho)}$.

A similar procedure can be applied for the antiparticle but selecting the normalized vectors $\hat{v}_{s\sigma} \in \mathcal{V}_{(\rho)}$ such that $\bar{v}_{s\sigma}\hat{v}_{s'\sigma'} = \delta_{ss'}\delta_{\sigma\sigma'}$ and

$$\bar{v}_{s\sigma}\rho[W(A, \vec{p})]\hat{v}_{s\sigma'} = [D_{\sigma\sigma'}^s(A, \vec{p})]^* \quad (47)$$

since the operators a and b must transform alike under isometries [21]. Moreover, from Eq. (37) we deduce that the vectors \hat{u} and \hat{v} must be orthogonal, $\bar{u}_{s\sigma}\hat{v}_{s'\sigma'} = \bar{v}_{s\sigma}\hat{u}_{s'\sigma'} = 0$. We remind the reader that the $SU(2)$ UIRs have the equivalence property $(D^s)^* = Y^s D^s (Y^s)^{-1}$ where the matrix Y^s has the elements $Y_{\sigma,\sigma'}^s = (-1)^{s-\sigma}\delta_{\sigma,-\sigma'}$ [5].

The conclusion is that the manifest CRs are equivalent to direct sums of Wigner UIRs with an arbitrary spin content defined by the vectors $\hat{u}_{s\sigma}$ and $\hat{v}_{s\sigma}$. For each spin s we meet the UIR $(\pm m, s)$ in the space $\mathcal{V}_s \subset \mathcal{V}_{(\rho)}$ of the linear UIR of the group $SU(2)$ generated by the matrices $S_i^{(s)}$.

The transformation (45) allows us to derive the generators of the UIRs in momentum rep. (denoted here by tilde) that are differential operator acting alike on the operators $a_{s\sigma}(\vec{p})$ and $b_{s\sigma}(\vec{p})$ seen as functions of \vec{p} . Thus for each UIR $(\pm m, s)$ we can write down the basis generators

$$\tilde{J}_i^{(s)} = -i\varepsilon_{ijk}p^j\partial_{p^k} + S_i^{(s)}, \quad (48)$$

$$\tilde{K}_i^{(s)} = iE\partial_{p^i} - \frac{p^i}{2E} + \frac{1}{E+m}\varepsilon_{ijk}p^jS_k^{(s)}. \quad (49)$$

With their help we derive the components of the Pauli-Lubanski operator

$$\tilde{W}_0^{(s)} = \vec{p} \cdot \vec{S}^{(s)}, \quad \tilde{W}_i^{(s)} = mS_i^{(s)} + \frac{p^i}{E+m}\vec{p} \cdot \vec{S}^{(s)}, \quad (50)$$

and we recover the well-known result $\tilde{C}_2^{(s)} = m^2(\vec{S}^{(s)})^2 \sim m^2s(s+1)$ [5].

Finally we stress that the Wigner theory determine completely the form of the covariant fields without using field equations. Thus in special relativity we have two symmetric equivalent procedures: (i) to start with the covariant field equation that gives the form of the covariant field determining thus its manifest CR, or (ii) to construct the Wigner covariant field and then to derive its field equation [21]. The typical example is the Dirac field on Minkowski spacetime [5, 24].

4 Covariant fields on de Sitter spacetime

The Wigner theory works only in local-Minkowskian manifold whose isometry group has a similar structure as the Poincaré one having an Abelian normal subgroup $T(4)$. Unfortunately the Abelian group $T(3)_P$ of the de Sitter isometry group is not a normal (or invariant) subgroup such that we cannot apply the Wigner method. Therefore, we must restrict ourselves to perform the general study of the de Sitter CRs only in the configuration space. We shall see that the theory the UIRs in momentum rep. may be constructed only when the structure of the covariant field is determined by a concrete field equations.

4.1 de Sitter isometries and Killing vectors

Let us start with the de Sitter spacetime (M, g) defined as the hyperboloid of radius $1/\omega$ ¹ in the five-dimensional flat spacetime (M^5, η^5) of

¹We denote by ω the Hubble de Sitter constant since H is reserved for the energy operator

coordinates z^A (labeled by the indices $A, B, \dots = 0, 1, 2, 3, 4$) and metric $\eta^5 = \text{diag}(1, -1, -1, -1, -1)$. The local charts $\{x\}$ can be introduced on (M, g) giving the set of functions $z^A(x)$ which solve the hyperboloid Eq.,

$$\eta_{AB}^5 z^A(x) z^B(x) = -\frac{1}{\omega^2}. \quad (51)$$

Here we use the chart $\{t, \vec{x}\}$ with the conformal time t and Cartesian spaces coordinates x^i defined by

$$\begin{aligned} z^0(x) &= -\frac{1}{2\omega^2 t} \left[1 - \omega^2(t^2 - \vec{x}^2) \right] \\ z^i(x) &= -\frac{1}{\omega t} x^i, \\ z^4(x) &= -\frac{1}{2\omega^2 t} \left[1 + \omega^2(t^2 - \vec{x}^2) \right] \end{aligned} \quad (52)$$

This chart covers the expanding part of M for $t \in (-\infty, 0)$ and $\vec{x} \in \mathbb{R}^3$ while the collapsing part is covered by a similar chart with $t > 0$. Both these charts have the conformal flat line element,

$$ds^2 = \eta_{AB}^5 dz^A(x) dz^B(x) = \frac{1}{\omega^2 t^2} (dt^2 - d\vec{x}^2). \quad (53)$$

In addition, we consider the local frames $\{t, \vec{x}; e\}$ of the diagonal gauge,

$$e_0^0 = -\omega t, \quad e_j^i = -\delta_j^i \omega t, \quad \hat{e}_0^0 = -\frac{1}{\omega t}, \quad \hat{e}_j^i = -\delta_j^i \frac{1}{\omega t}. \quad (54)$$

The gauge group $G(\eta^5) = SO(1, 4)$ is the isometry group of M , since its transformations, $z \rightarrow \mathfrak{g}z$, $\mathfrak{g} \in SO(1, 4)$, leave the Eq. (51) invariant. Its universal covering group $\text{Spin}(1, 4) = Sp(2, 2)$ is not involved directly in our construction since the spinor CRs are induced by the spinor reps. of its subgroup $SL(2, \mathbb{C})$. Therefore, we can restrict ourselves to the group $SO(1, 4)$ for which we adopt the parametrization

$$\mathfrak{g}(\xi) = \exp \left(-\frac{i}{2} \xi^{AB} \mathfrak{S}_{AB} \right) \in SO(1, 4) \quad (55)$$

with skew-symmetric parameters, $\xi^{AB} = -\xi^{BA}$, and the covariant generators \mathfrak{S}_{AB} of the fundamental rep. of the $so(1, 4)$ algebra carried by M^5 . These generators have the matrix elements,

$$(\mathfrak{S}_{AB})_{\cdot D}^{\cdot C} = i \left(\delta_A^C \eta_{BD}^5 - \delta_B^C \eta_{AD}^5 \right). \quad (56)$$

The principal $so(1, 4)$ basis-generators with physical meaning [17] are the energy $\mathfrak{H} = \omega \mathfrak{S}_{04}$, angular momentum $\mathfrak{J}_k = \frac{1}{2} \varepsilon_{kij} \mathfrak{S}_{ij}$, Lorentz boosts

$\mathfrak{K}_i = \mathfrak{S}_{0i}$, and the Runge-Lenz-type vector $\mathfrak{R}_i = \mathfrak{S}_{i4}$. In addition, it is convenient to introduce the momentum $\mathfrak{P}_i = -\omega(\mathfrak{R}_i + \mathfrak{K}_i)$ and its dual $\mathfrak{Q}_i = \omega(\mathfrak{R}_i - \mathfrak{K}_i)$ which are nilpotent matrices (i. e. $(\mathfrak{P}_i)^3 = (\mathfrak{Q}_i)^3 = 0$) of two Abelian three-dimensional subalgebras, $t(3)_P$ and respectively $t(3)_Q$ generating the Abelian subgroups $T(3)_P$ and $T(3)_Q$. Among all these generators we may chose different bases of the algebra $so(1,4)$ as, for example, the basis $\{\mathfrak{H}, \mathfrak{P}_i, \mathfrak{Q}_i, \mathfrak{J}_i\}$ or the Poincaré-type one, $\{\mathfrak{H}, \mathfrak{P}_i, \mathfrak{J}_i, \mathfrak{K}_i\}$. We note that the four-dimensional restriction of the $so(1,3)$ subalegra generate the vector rep. of the group L_+^\uparrow .

Using these generators we can derive the $SO(1,4)$ isometries, $\phi_{\mathfrak{g}}$, defined as

$$z[\phi_{\mathfrak{g}}(x)] = \mathfrak{g} z(x). \quad (57)$$

The transformations $\mathfrak{g} \in SO(3) \subset SO(4,1)$ generated by \mathfrak{J}_i , are simple rotations of z^i and x^i which transform alike since this symmetry is global. The transformations generated by \mathfrak{H} ,

$$\begin{aligned} \exp(-i\xi\mathfrak{H}) : \quad & z^0 \rightarrow z^0 \cosh \alpha - z^4 \sinh \alpha \\ & z^i \rightarrow z^i \\ & z^4 \rightarrow -z^0 \sinh \alpha + z^4 \cosh \alpha \end{aligned} \quad (58)$$

whith $\alpha = \omega\xi$, produce the dilatations $t \rightarrow t e^\alpha$ and $x^i \rightarrow x^i e^\alpha$, while the $T(3)_P$ transformations

$$\begin{aligned} \exp(-i\xi^i \mathfrak{P}_i) : \quad & z^0 \rightarrow z^0 + \omega \vec{\xi} \cdot \vec{z} + \frac{1}{2} \omega^2 \vec{\xi}^2 (z^0 + z^4) \\ & z^i \rightarrow z^i + \omega \xi^i (z^0 + z^4) \\ & z^4 \rightarrow z^4 - \omega \vec{\xi} \cdot \vec{z} - \frac{1}{2} \omega^2 \vec{\xi}^2 (z^0 + z^4) \end{aligned} \quad (59)$$

give rise to the space translations $x^i \rightarrow x^i + \xi^i$ at fixed t . More interesting are the $T(3)_Q$ transformations generated by \mathfrak{Q}_i/ω ,

$$\begin{aligned} \exp(-i\xi^i \mathfrak{Q}_i/\omega) : \quad & z^0 \rightarrow z^0 - \vec{\xi} \cdot \vec{z} + \frac{1}{2} \vec{\xi}^2 (z^0 - z^4) \\ & z^i \rightarrow z^i - \xi^i (z^0 - z^4) \\ & z^4 \rightarrow z^4 - \vec{\xi} \cdot \vec{z} + \frac{1}{2} \vec{\xi}^2 (z^0 - z^4) \end{aligned} \quad (60)$$

which lead to the isometries

$$t \rightarrow \frac{t}{1 - 2\omega \vec{\xi} \cdot \vec{x} - \omega^2 \vec{\xi}^2 (t^2 - \vec{x}^2)} \quad (61)$$

$$x^i \rightarrow \frac{x^i + \omega \xi^i (t^2 - \vec{x}^2)}{1 - 2\omega \vec{\xi} \cdot \vec{x} - \omega^2 \vec{\xi}^2 (t^2 - \vec{x}^2)}. \quad (62)$$

We observe that $z^0 + z^4 = -\frac{1}{\omega^2 t}$ is invariant under translations (59), fixing the value of t , while $z^0 - z^4 = \frac{t^2 - \vec{x}^2}{t}$ is left unchanged by the $t(3)_Q$ transformations (60).

The orbital basis-generators of the natural rep. of the $s(M)$ algebra (carried by the space of the scalar functions over M^5) have the standard form

$$L_{AB}^5 = i \left[\eta_{AC}^5 z^C \partial_B - \eta_{BC}^5 z^C \partial_A \right] = -i K_{(AB)}^C \partial_C \quad (63)$$

which allows us to derive the corresponding Killing vectors of (M, g) , $k_{(AB)}$, using the identities $k_{(AB)\mu} dx^\mu = K_{(AB)C} dz^C$. Thus we obtain the following components of the Killing vectors:

$$k_{(04)}^0 = t, \quad k_{(04)}^i = x^i, \quad k_{(0i)}^0 = k_{(4i)}^0 = \omega t x^i \quad (64)$$

$$k_{(0i)}^j = \omega x^i x^j + \delta_i^j \frac{1}{2\omega} [\omega^2 (t^2 - \vec{x}^2) - 1] \quad (65)$$

$$k_{(4i)}^j = \omega x^i x^j + \delta_i^j \frac{1}{2\omega} [\omega^2 (t^2 - \vec{x}^2) + 1] \quad (66)$$

$$k_{(ij)}^k = \delta_j^k x^i - \delta_i^k x^j. \quad (67)$$

4.2 Generators of induced CRs

In the covariant parametrization of the $sp(2, 2)$ algebra, adopted here, the generators $X_{(AB)}^{(\rho)}$ corresponding to the Killing vectors $k_{(AB)}$ result from Eq. (10) and the functions (8) with the new labels $a \rightarrow (AB)$. Using then the components (64) - (67) and the tetrad-gauge (54) of the chart $\{t, \vec{x}\}$, after a little calculation, we find the form of the $sl(2, \mathbb{C})$ generators of the CRs induced by the rep. ρ , denoting from now by $\rho(S) = S^{(\rho)}$.

These operators are the energy (or Hamiltonian) H , total angular momentum \vec{J} , generators of the Lorentz boosts \vec{K} , and Runge-Lenz type vector \vec{R} , whose components read [17],

$$H = \omega X_{(04)}^{(\rho)} = -i\omega(t\partial_t + x^i\partial_i), \quad (68)$$

$$J_i^{(\rho)} = \frac{1}{2} \varepsilon_{ijk} X_{(jk)}^{(\rho)} = -i\varepsilon_{ijk} x^j \partial_k + S_i^{(\rho)}, \quad S_i^{(\rho)} = \frac{1}{2} \varepsilon_{ijk} S_{jk}^{(\rho)}, \quad (69)$$

$$K_i^{(\rho)} = X_{(0i)}^{(\rho)} = ix^i H + \frac{i}{2\omega} [1 + \omega^2(\vec{x}^2 - t^2)] \partial_i - \omega t S_{0i}^{(\rho)} + \omega S_{ij}^{(\rho)} x^j, \quad (70)$$

$$R_i^{(\rho)} = X_{(i4)}^{(\rho)} = -K_i^{(\rho)} + \frac{1}{\omega} i\partial_i. \quad (71)$$

These generators form the basis $\{H, J_i^{(\rho)}, K_i^{(\rho)}, R_i^{(\rho)}\}$ of the induced CR of the $sp(2, 2)$ algebra with the following commutation rules:

$$[J_i^{(\rho)}, J_j^{(\rho)}] = i\varepsilon_{ijk} J_k^{(\rho)}, \quad [J_i^{(\rho)}, R_j^{(\rho)}] = i\varepsilon_{ijk} R_k^{(\rho)}, \quad (72)$$

$$[J_i^{(\rho)}, K_j^{(\rho)}] = i\varepsilon_{ijk}K_k^{(\rho)}, \quad [R_i^{(\rho)}, R_j^{(\rho)}] = i\varepsilon_{ijk}J_k^{(\rho)}, \quad (73)$$

$$[K_i^{(\rho)}, K_j^{(\rho)}] = -i\varepsilon_{ijk}J_k^{(\rho)}, \quad [R_i^{(\rho)}, K_j^{(\rho)}] = \frac{i}{\omega}\delta_{ij}H, \quad (74)$$

and

$$[H, J_i^{(\rho)}] = 0, \quad [H, K_i^{(\rho)}] = i\omega R_i^{(\rho)}, \quad [H, R_i^{(\rho)}] = i\omega K_i^{(\rho)}. \quad (75)$$

In some applications it is useful to replace the operators $\vec{K}^{(\rho)}$ and $\vec{R}^{(\rho)}$ by the Abelian ones, i. e. the momentum operator \vec{P} and its dual $\vec{Q}^{(\rho)}$, whose components are defined as

$$P_i = -\omega(R_i^{(\rho)} + K_i^{(\rho)}) = -i\partial_i, \quad Q_i^{(\rho)} = \omega(R_i^{(\rho)} - K_i^{(\rho)}). \quad (76)$$

Then the basis is $\{H, P_i, Q_i^{(\rho)}, J_i^{(\rho)}\}$ with the new commutators

$$[H, P_i] = i\omega P_i, \quad [H, Q_i^{(\rho)}] = -i\omega Q_i^{(\rho)}, \quad (77)$$

$$[J_i^{(\rho)}, P_j] = i\varepsilon_{ijk}P_k, \quad [J_i^{(\rho)}, Q_j^{(\rho)}] = i\varepsilon_{ijk}Q_k^{(\rho)}, \quad (78)$$

$$[Q_i^{(\rho)}, P_j] = -2i\omega\delta_{ij}H - 2i\omega^2\varepsilon_{ijk}J_k^{(\rho)}, \quad (79)$$

$$[Q_i^{(\rho)}, Q_j^{(\rho)}] = [P_i, P_j] = 0. \quad (80)$$

Another basis is of the Poincaré type being formed by $\{H, P_i, J_i^{(\rho)}, K_i^{(\rho)}\}$. This has the commutation rules given by Eqs. (72a), (73a), (74a), (77a), (78a) and $[K_j^{(\rho)}, P_i] = i\delta_{ij}H - i\omega\varepsilon_{ijk}J_k^{(\rho)}$, while the commutator (75b) has to be rewritten as $[K_i^{(\rho)}, H] = iP_i + i\omega K_i^{(\rho)}$.

The last two bases bring together the conserved energy (68) and momentum (76a) which are the only genuine orbital operators, independent on ρ . What is specific for the de Sitter symmetry is that these operators cannot be put simultaneously in diagonal form since, according to Eq. (77a), they do not commute to each other.

4.3 Casimir operators

The first invariant of the CR $T^{(\rho)}$ is the quadratic Casimir operator

$$\mathcal{C}_1^{(\rho)} = -\omega^2 \frac{1}{2} X_{(AB)}^{(\rho)} X^{(\rho)(AB)} \quad (81)$$

$$= H^2 - \omega^2 (\vec{J}^{(\rho)} \cdot \vec{J}^{(\rho)} + \vec{R}^{(\rho)} \cdot \vec{R}^{(\rho)} - \vec{K}^{(\rho)} \cdot \vec{K}^{(\rho)}) \quad (82)$$

$$= H^2 + 3i\omega H - \vec{Q}^{(\rho)} \cdot \vec{P} - \omega^2 \vec{J}^{(\rho)} \cdot \vec{J}^{(\rho)}. \quad (83)$$

which can be calculated according to Eqs. (69)-(68) and (76). After a few manipulation we obtain its definitive expression

$$\mathcal{C}_1^{(\rho)} = \mathcal{E}_{KG} + 2i\omega e^{-\omega t} S_{0i}^{(\rho)} \partial_i - \omega^2 (\vec{S}^{(\rho)})^2, \quad (84)$$

depending on the Klein-Gordon operator of the scalar field,

$$\mathcal{E}_{KG} = -\partial_t^2 - 3\omega \partial_t + e^{-2\omega t} \Delta, \quad \Delta = \vec{\partial}^2. \quad (85)$$

The second Casimir operator,

$$\mathcal{C}_2^{(\rho)} = -\eta_{AB}^5 W^{(\rho)A} W^{(\rho)B}, \quad (86)$$

is written with the help of the five-dimensional vector-operator $W^{(\rho)}$ whose components read [10]

$$W^{(\rho)A} = \frac{1}{8} \omega \varepsilon^{ABCDE} X_{(BC)}^{(\rho)} X_{(DE)}^{(\rho)}, \quad (87)$$

where $\varepsilon^{01234} = 1$ and the factor ω assures the correct flat limit. After a little calculation we obtain the concrete form of these components,

$$W_0^{(\rho)} = \omega \vec{J}^{(\rho)} \cdot \vec{R}^{(\rho)}, \quad (88)$$

$$W_i^{(\rho)} = H J_i^{(\rho)} + \omega \varepsilon_{ijk} K_j^{(\rho)} R_k^{(\rho)}, \quad (89)$$

$$W_4^{(\rho)} = -\omega \vec{J}^{(\rho)} \cdot \vec{K}^{(\rho)}, \quad (90)$$

which indicate that $W^{(\rho)}$ plays an important role in theories with spin, similar to that of the Pauli-Lubanski operator (32) of the Poincaré symmetry. For example, the helicity operator is now $W_0^{(\rho)} - W_4^{(\rho)} = S_i^{(\rho)} P_i$.

Replacing then the components (88)-(90) in Eq. (86) we are faced with a complicated calculation but which can be performed using algebraic codes on computer. Thus we obtain the closed form of the second Casimir operator,

$$\begin{aligned} \mathcal{C}_2^{(\rho)} = & -\omega^2 (\vec{S}^{(\rho)})^2 (t^2 \partial_t^2 - 2t \partial_t + 2) + 2\omega^2 t^2 (i S_{0k}^{(\rho)} - \varepsilon_{ijk} S_i^{(\rho)} S_{0j}^{(\rho)}) \partial_k \partial_t \\ & + \omega t \left[(\vec{S}_0^{(\rho)})^2 \Delta - (S_i^{(\rho)} S_j^{(\rho)} + S_{0i}^{(\rho)} S_{0j}^{(\rho)}) \partial_i \partial_j \right] \\ & - 2i\omega^2 t (S_i^{(\rho)} S_k^{(\rho)} S_{0i}^{(\rho)} + S_{0k}^{(\rho)}) \partial_k. \end{aligned} \quad (91)$$

In the case of fields with unique spin s we must select the reps. $\rho(s) = (s, 0) \oplus (0, s)$, for which we have to replace $S_{0i}^{\rho(s)} = \pm i S_i^{\rho(s)}$ in Eq. (91) finding the remarkable identity

$$\mathcal{C}_2^{\rho(s)} = \mathcal{C}_1^{\rho(s)} (\vec{S}^{\rho(s)})^2 - 2\omega^2 (\vec{S}^{\rho(s)})^2 + \omega^2 [(\vec{S}^{\rho(s)})^2]^2. \quad (92)$$

It is interesting to look for the invariants of the particles at rest in the chart $\{t, \vec{x}\}$. These have the vanishing momentum ($P_i \sim 0$) so that H acts as $i\partial_t$ and, therefore, it can be put in diagonal form its eigenvalues being just the rest energies, E_0 . Then, for each subspace $\mathcal{V}_s \subset \mathcal{V}_{(\rho)}$ of given spin, s , we obtain the eigenvalues of the first Casimir operator,

$$\mathcal{C}_1^{(\rho)} \sim E_0^2 + 3i\omega E_0 - \omega^2 s(s+1), \quad (93)$$

using Eqs. (84) and (85) while those of the second Casimir operator,

$$\mathcal{C}_2^{(\rho)} \sim s(s+1)(E_0^2 + 3i\omega E_0 - 2\omega^2), \quad (94)$$

result from Eq. (91). These eigenvalues are real numbers so that the rest energies, $E_0 = \Re E_0 - \frac{3i\omega}{2}$, must be complex numbers whose imaginary parts are due to the decay produced by the de Sitter expansion.

The above results indicate that the induced CRs are reducible to direct sums of UIRs of the principal series [6], (p, q) , with $p = s$ and $q(1-q) = \frac{1}{\omega^2}(\Re E_0)^2 + \frac{1}{4}$. What is new here is that we meet only one type of UIRs which are completely determined by the rest energy and the spin defined as in special relativity. In the flat limit these UIRs tend to the corresponding Wigner ones.

Finally, we specify that the physical interpretation adopted here is correct since in the flat limit we recover the Poincaré generators as given in the Appendix A. We observe that the generators (69) are independent on ω having the same form as in the Minkowski case, $J_k^{(\rho)} = \hat{J}_k^{(\rho)}$. The other generators have the limits

$$\lim_{\omega \rightarrow 0} H = \hat{H} = i\partial_t, \quad \lim_{\omega \rightarrow 0} (\omega R_i^{(\rho)}) = -\hat{P}_i = i\partial_i, \quad \lim_{\omega \rightarrow 0} K_i^{(\rho)} = \hat{K}_i^{(\rho)}, \quad (95)$$

which means that the basis $\{H, P_i, J_i^{(\rho)}, K_i^{(\rho)}\}$ of the algebra $s(M) = sp(2, 2)$ tends to the basis $\{\hat{H}, \hat{P}_i, \hat{J}_i^{(\rho)}, \hat{K}_i^{(\rho)}\}$ of the $s(M_0)$ algebra when $\omega \rightarrow 0$. Moreover, the Pauli-Lubanski operator (32) is the flat limit of the five-dimensional vector-operator (87) since

$$\lim_{\omega \rightarrow 0} W_0^{(\rho)} = \hat{W}_0^{(\rho)}, \quad \lim_{\omega \rightarrow 0} W_i^{(\rho)} = \hat{W}_i^{(\rho)}, \quad \lim_{\omega \rightarrow 0} W_4^{(\rho)} = 0. \quad (96)$$

Under such circumstances the limits of our invariants read

$$\lim_{\omega \rightarrow 0} \mathcal{C}_1^{(\rho)} = \hat{\mathcal{C}}_1 = \hat{P}^2, \quad \lim_{\omega \rightarrow 0} \mathcal{C}_2^{(\rho)} = \hat{\mathcal{C}}_2^{(\rho)}, \quad (97)$$

indicating that their physical meaning may be related to the mass and spin of the matter fields in a similar manner as in special relativity.

5 The Dirac field on de Sitter spacetimes

In the absence of a strong theory like the Wigner one, we must study the CR-UIR equivalence resorting to the covariant field equations able to give us the structure of the covariant field. Then, bearing in mind that the de Sitter UIRs are well-studied [6, 7], we can establish the CR-UIR equivalence by studying the CR and UIR Casimir operators in configurations and momentum rep..

In what follows we concentrate on the Dirac equation on the de Sitter spacetime since this is the only Eq. on this background giving the natural rest energy $\Re E_0 = m$ [17].

5.1 Invariants of the spinor CR

In the frame $\{t, \vec{x}; e\}$ introduced above the free Dirac equation takes the form [15],

$$(\mathcal{E}_D - m)\psi(x) = \left[-i\omega t \left(\gamma^0 \partial_t + \gamma^i \partial_i \right) + \frac{3i\omega}{2} \gamma^0 - m \right] \psi(x) = 0, \quad (98)$$

depending on the point-independent Dirac matrices $\gamma^{\hat{\mu}}$ that satisfy the well-known algebra $\{\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}\} = 2\eta^{\hat{\alpha}\hat{\beta}}$ giving rise to the basis-generators $S^{(\rho_s)\hat{\alpha}\hat{\beta}} = \frac{i}{4}[\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}]$ of the spinor rep. $\rho_s = \rho(\frac{1}{2}) = (\frac{1}{2}, 0) \otimes (0, \frac{1}{2})$ of the group $\hat{G} = SL(2, \mathbb{C})$ that induces the spinor CR [1, 15, 16].

Eq. (98) can be analytically solved either in momentum or energy bases with correct orthonormalization and completeness properties [15, 16] with respect to the relativistic scalar product

$$\langle \psi, \psi' \rangle = \int d^3x (-\omega t)^{-3} \overline{\psi}(t, \vec{x}) \gamma^0 \psi'(t, \vec{x}). \quad (99)$$

The mode expansion in the spin-momentum rep. [16],

$$\psi(t, \vec{x}) = \int d^3p \sum_{\sigma} \left[U_{\vec{p}, \sigma}(x) a(\vec{p}, \sigma) + V_{\vec{p}, \sigma}(x) b^{\dagger}(\vec{p}, \sigma) \right], \quad (100)$$

is written in terms of the field operators, a and b (satisfying canonical anti-commutation rules), and the particle and antiparticle fundamental spinors of momentum \vec{p} (with $p = |\vec{p}|$) and polarization $\sigma = \pm \frac{1}{2}$,

$$U_{\vec{p}, \sigma}(t, \vec{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} u_{\vec{p}, \sigma}(t) e^{i\vec{p} \cdot \vec{x}}, \quad V_{\vec{p}, \sigma}(t, \vec{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} v_{\vec{p}, \sigma}(t) e^{-i\vec{p} \cdot \vec{x}} \quad (101)$$

whose time-dependent terms have the form [16, 18]

$$u_{\vec{p},\sigma}(t) = \frac{i}{2} \left(\frac{\pi p}{\omega} \right)^{\frac{1}{2}} (\omega t)^2 \begin{pmatrix} e^{\frac{1}{2}\pi\mu} H_{\nu_-}^{(1)}(-pt) \xi_\sigma \\ e^{-\frac{1}{2}\pi\mu} H_{\nu_+}^{(1)}(-pt) \frac{\vec{\sigma} \cdot \vec{p}}{p} \xi_\sigma \end{pmatrix}, \quad (102)$$

$$v_{\vec{p},\sigma}(t) = \frac{i}{2} \left(\frac{\pi p}{\omega} \right)^{\frac{1}{2}} (\omega t)^2 \begin{pmatrix} e^{-\frac{1}{2}\pi\mu} H_{\nu_-}^{(2)}(-pt) \frac{\vec{\sigma} \cdot \vec{p}}{p} \eta_\sigma \\ e^{\frac{1}{2}\pi\mu} H_{\nu_+}^{(2)}(-pt) \eta_\sigma \end{pmatrix}, \quad (103)$$

in the standard rep. of the Dirac matrices (with diagonal γ^0) and a fixed vacuum of the Bunch-Davies type [18]. Obviously, the notation σ_i stands for the Pauli matrices while the point-independent Pauli spinors ξ_σ and $\eta_\sigma = i\sigma_2(\xi_\sigma)^*$ are normalized as $\xi_\sigma^+ \xi_{\sigma'} = \eta_\sigma^+ \eta_{\sigma'} = \delta_{\sigma\sigma'}$ [16]. The terms giving the time modulation depend on the Hankel functions $H_{\nu_\pm}^{(1,2)}$ of indices

$$\nu_\pm = \frac{1}{2} \pm i\mu, \quad \mu = \frac{m}{\omega}. \quad (104)$$

Based on their properties (presented in Appendix B) we deduce

$$u_{\vec{p},\sigma}^+(t) u_{\vec{p},\sigma}(t) = v_{\vec{p},\sigma}^+(t) v_{\vec{p},\sigma}(t) = (-\omega t)^3 \quad (105)$$

obtaining the orthonormalization relations [15]

$$\langle U_{\vec{p},\sigma}, U_{\vec{p}',\sigma'} \rangle = \langle V_{\vec{p},\sigma}, V_{\vec{p}',\sigma'} \rangle = \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}'), \quad (106)$$

$$\langle U_{\vec{p},\sigma}, V_{\vec{p}',\sigma'} \rangle = \langle V_{\vec{p},\sigma}, U_{\vec{p}',\sigma'} \rangle = 0, \quad (107)$$

that yield the useful inversion formulas,

$$a(\vec{p}, \sigma) = \langle U_{\vec{p},\sigma}, \psi \rangle, \quad b(\vec{p}, \sigma) = \langle \psi, V_{\vec{p},\sigma} \rangle. \quad (108)$$

Moreover, it is not hard to verify that these spinors are charge-conjugated to each other,

$$V_{\vec{p},\sigma} = (U_{\vec{p},\sigma})^c = \mathcal{C}(\overline{U}_{\vec{p},\sigma})^T, \quad \mathcal{C} = i\gamma^2\gamma^0, \quad (109)$$

and represent a *complete* system of solutions in the sense that [15]

$$\int d^3p \sum_\sigma \left[U_{\vec{p},\sigma}(t, \vec{x}) U_{\vec{p},\sigma}^+(t, \vec{x}') + V_{\vec{p},\sigma}(t, \vec{x}) V_{\vec{p},\sigma}^+(t, \vec{x}') \right] = e^{-3\omega t} \delta^3(\vec{x} - \vec{x}'). \quad (110)$$

The Dirac field transforms under isometries $x \rightarrow x' = \phi_{\mathfrak{g}}(x)$ (with $\mathfrak{g} \in I(M)$) according to the CR $T_{\mathfrak{g}} : \psi(x) \rightarrow (T_{\mathfrak{g}}\psi)(x') = A_{\mathfrak{g}}(x)\psi(x)$ whose generators are given by Eqs. (68) - (71) where now $\rho = \rho_s$. Then, according to Eqs. (84) and (98) we obtain the identity

$$\mathcal{C}_1^{(\rho_s)} = \mathcal{E}_D^2 + \frac{3}{2} \omega^2 \mathbf{1}_{4 \times 4} \sim m^2 + \frac{3}{2} \omega^2. \quad (111)$$

This result and Eq. (93) yield the rest energy of the Dirac field,

$$E_0 = -\frac{3i\omega}{2} \pm m, \quad (112)$$

which has a natural simple form where the decay (first) term is added to the usual rest energy of special relativity. A similar result can be obtained by solving the Dirac equation with vanishing momentum.

The second invariant results from Eqs. (92) and (111) if we take into account that $(\vec{S}^{(\rho_s)})^2 = \frac{3}{4} \mathbf{1}_{4 \times 4}$. Thus we find

$$\mathcal{C}_2^{(\rho_s)} = \frac{3}{4} \mathcal{E}_D^2 + \frac{3}{16} \omega^2 \mathbf{1}_{4 \times 4} \sim \frac{3}{4} \left(m^2 + \frac{1}{4} \omega^2 \right) = \omega^2 s(s+1) \nu_+ \nu_-, \quad (113)$$

where $s = \frac{1}{2}$ is the spin and $\nu_{\pm} = \frac{1}{2} \pm i \frac{m}{\omega}$ are the indices of the Hankel functions giving the time modulation of the Dirac spinors of the momentum basis [15].

These invariants define the UIRs that in the flat limit become Wigner's UIRs $(\pm m, \frac{1}{2})$ since

$$\lim_{\omega \rightarrow 0} \mathcal{C}_1^{(\rho_s)} \sim m^2, \quad \lim_{\omega \rightarrow 0} \mathcal{C}_2^{(\rho_s)} \sim \frac{3}{4} m^2. \quad (114)$$

5.2 Invariants of UIRs in momentum representation

The inversion formulas (108) allow us to write the transformation rules in momentum rep. as

$$\begin{aligned} (T_{\mathfrak{g}} a)(\vec{p}, \sigma) &= \left\langle U_{\vec{p}, \sigma}, [\rho_s(A_{\mathfrak{g}}) \psi] \circ \phi_{\mathfrak{g}}^{-1} \right\rangle \\ &= \int d^3 p' \sum_{\sigma'} \left\langle U_{\vec{p}, \sigma}, [\rho_s(A_{\mathfrak{g}}) U_{\vec{p}', \sigma'}] \circ \phi_{\mathfrak{g}}^{-1} \right\rangle a(\vec{p}', \sigma'), \end{aligned} \quad (115)$$

$$\begin{aligned} (T_{\mathfrak{g}} b)(\vec{p}, \sigma) &= \left\langle [\rho_s(A_{\mathfrak{g}}) \psi] \circ \phi_{\mathfrak{g}}^{-1}, V_{\vec{p}, \sigma} \right\rangle \\ &= \int d^3 p' \sum_{\sigma'} \left\langle [\rho_s(A_{\mathfrak{g}}) V_{\vec{p}', \sigma'}] \circ \phi_{\mathfrak{g}}^{-1}, V_{\vec{p}, \sigma} \right\rangle b(\vec{p}', \sigma'), \end{aligned} \quad (116)$$

but, unfortunately, these scalar products are complicated integrals that cannot be solved in the general case.

Nevertheless, one can prove that for the particular isometries $\mathfrak{g} = \mathfrak{g}(\vec{\omega}) \mathfrak{g}(\vec{a})$ of the Euclidean subgroup $E(3)$, formed by a rotation $\mathfrak{g}(\vec{\omega}) \in SO(3)$ and a translation $\mathfrak{g}(\vec{a}) \in T(3)_P$, we find the linear transformations,

$$(T_{\mathfrak{g}} a)(\vec{p}, \sigma) = D_{\sigma\sigma'}^{(\frac{1}{2})}(\vec{\omega}) a(\vec{p}', \sigma') e^{i\vec{a} \cdot \vec{p}'}, \quad \vec{p}' = R^{-1}(\vec{\omega}) \vec{p}, \quad (117)$$

that are somewhat similar with those of the Wigner theory. However, there are isometries that cannot be brought in a such simple form as, for example, those of the adjoint Abelian subgroup $T(3)_Q$. Therefore, we must abandon the group transformations focusing on the generators of the corresponding Lie algebras in momentum rep..

Any self-adjoint generator X of the spinor rep. of the group $S(M)$ gives rise to a *conserved* one-particle operator of the QFT,

$$\mathbf{X} =: \langle \psi, X\psi \rangle := \mathbf{X}^{(+)} + \mathbf{X}^{(-)} = \int d^3p \left[\alpha^\dagger(\vec{p}) \tilde{X}^{(+)} \alpha(\vec{p}) + \beta^\dagger(\vec{p}) \tilde{X}^{(-)} \beta(\vec{p}) \right], \quad (118)$$

calculated respecting the normal ordering of the operator products [23]. The operators $\tilde{X}^{(\pm)}$ are the generators of CRs in momentum rep. acting on the operator valued Pauli spinors,

$$\alpha(\vec{p}) = \begin{pmatrix} a(\vec{p}, \frac{1}{2}) \\ a(\vec{p}, -\frac{1}{2}) \end{pmatrix}, \quad \beta(\vec{p}) = \begin{pmatrix} b(\vec{p}, \frac{1}{2}) \\ b(\vec{p}, -\frac{1}{2}) \end{pmatrix}. \quad (119)$$

As observed in Ref. [13], the straightforward method for finding the structure of these operators is to evaluate the entire expression (118) by using the form (100) where the field operators a and b satisfy the *canonical anti-commutation rules* [13, 15].

For this purpose we consider several identities written with the notation $\partial_{p_i} = \frac{\partial}{\partial p_i}$ as

$$\begin{aligned} H U_{\vec{p},\sigma}(t, \vec{x}) &= -i\omega \left(p^i \partial_{p_i} + \frac{3}{2} \right) U_{\vec{p},\sigma}(t, \vec{x}), \\ H V_{\vec{p},\sigma}(t, \vec{x}) &= -i\omega \left(p^i \partial_{p_i} + \frac{3}{2} \right) V_{\vec{p},\sigma}(t, \vec{x}), \end{aligned}$$

that help us to eliminate some multiplicative operators and the time derivative when we inverse the Fourier transform. Furthermore, by applying the Green theorem and calculating on computer terms of the form $u_{\vec{p},\sigma}^+(t)F(t, p_i)u_{\vec{p},\sigma}(t)$, $v_{\vec{p},\sigma}^+(t)F(t, p_i)v_{\vec{p},\sigma}(t)$, ..., etc., we find two *identical* reps. whose basis generators read, $\tilde{P}_i^{(\pm)} = \tilde{P}_i = p_i$ and

$$\tilde{H}^{(\pm)} = \omega \tilde{X}_{(04)}^{(\pm)} = i\omega \left(p_i \partial_{p_i} + \frac{3}{2} \right) \quad (120)$$

$$\tilde{J}_i^{(\pm)} = \frac{1}{2} \varepsilon_{ijk} \tilde{X}_{(jk)}^{(\pm)} = -i\varepsilon_{ijk} p_j \partial_{p_k} + \frac{1}{2} \sigma_i \quad (121)$$

$$\begin{aligned} \tilde{K}_i^{(\pm)} &= \tilde{X}_{(0i)}^{(\pm)} = i\tilde{H}^{(\pm)} \partial_{p_i} + \frac{\omega}{2} p_i \Delta_p - p_i \frac{\vec{p}^2 + m^2}{2\omega \vec{p}^2} \\ &\quad + \frac{1}{2} \varepsilon_{ijk} \left(i\omega \partial_{p_j} - p_j \frac{m}{2\vec{p}^2} \right) \sigma_k \end{aligned} \quad (122)$$

$$\tilde{R}_i^{(\pm)} = \tilde{X}_{(i4)}^{(\pm)} = -\tilde{K}_i^{(\pm)} - \frac{1}{\omega} \tilde{P}_i, \quad (123)$$

where $\Delta_p = \partial_{p_i} \partial_{p_i}$. These basis generators satisfy the specific $sp(2, 2)$ commutation rules of the form (72)-(75). Moreover, it is not difficult to verify that these are Hermitian operators with respect to the scalar products of the momentum rep.

$$\langle \alpha, \alpha' \rangle = \int d^3 p \alpha^\dagger(\vec{p}) \tilde{\alpha}(\vec{p}), \quad \langle \beta, \beta' \rangle = \int d^3 p \beta^\dagger(\vec{p}) \tilde{\beta}(\vec{p}). \quad (124)$$

Therefore, we can conclude that these operators generate a pair of *unitary* reps. of the group $S(M)$.

Since all the UIRs of the group $S(M)$ are classified [6], we can study the equivalence and reducibility of these reps. simply by calculating the Casimir operators in momentum rep.. By using the same definitions as in the configurations rep. we write the first Casimir operator as,

$$\tilde{\mathcal{C}}_1 = -\frac{1}{2} \omega^2 \tilde{X}_{(AB)} \tilde{X}^{(AB)}, \quad (125)$$

while the second one,

$$\tilde{\mathcal{C}}_2 = -\eta_{AB}^5 \tilde{W}^A \tilde{W}^B, \quad \tilde{W}^A = \frac{1}{8} \omega \varepsilon^{ABCDE} \tilde{X}_{(BC)} \tilde{X}_{(DE)}, \quad (126)$$

is written in terms of the Pauli-Lubanski operator of components \tilde{W}^A . After performing the calculation on computer we find

$$\begin{aligned} \tilde{W}_0^{(\pm)} &= \frac{\omega}{4} (\vec{\sigma} \cdot \vec{p}) \Delta_p + \frac{\omega \nu_-}{2} \vec{\sigma} \cdot \vec{\partial}_p + \frac{im}{2p^2} (\vec{\sigma} \cdot \vec{p}) \vec{p} \cdot \vec{\partial}_p \\ &\quad + \frac{m^2 - \vec{p}^2 + 2i\omega m}{4\vec{p}^2 \omega} \vec{\sigma} \cdot \vec{p}, \end{aligned} \quad (127)$$

$$\tilde{W}_i^{(\pm)} = \frac{i}{2} (\vec{\sigma} \cdot \vec{p}) \partial_{p_i} - \frac{i\nu_-}{2\vec{p}^2} \sigma_i - \frac{m}{2\omega \vec{p}^2} (\vec{\sigma} \cdot \vec{p}) p_i, \quad (128)$$

$$\tilde{W}_4^{(\pm)} = \tilde{W}_0^{(\pm)} + \frac{1}{2\omega} \vec{\sigma} \cdot \vec{p}. \quad (129)$$

With these preparation we obtain the Casimir operators (125) and (126) as

$$\tilde{\mathcal{C}}_1^{(\pm)} = \omega^2 [-s(s+1) - (q+1)(q-2)] = m^2 + \frac{3\omega^2}{2}, \quad (130)$$

$$\begin{aligned} \tilde{\mathcal{C}}_2^{(\pm)} &= \omega^2 [-s(s+1)q(q-1)] \\ &= \omega^2 s(s+1)\nu_+\nu_- = \frac{3}{4} \left(m^2 + \frac{\omega^2}{4} \right), \end{aligned} \quad (131)$$

recovering thus the Casimir eigenvalues (111) and (113) obtained in configurations.

5.3 CR-UIR equivalence

The above result shows that the identical spinor reps. we obtained here are UIRs of the principal series corresponding to the canonical labels (s, q) with $s = \frac{1}{2}$ and $q = \nu_{\pm}$. In other words the spinor CR of the Dirac theory is equivalent with the orthogonal sum of the equivalent UIRs of the particle and antiparticle sectors. This suggests that the UIRs (s, ν_{\pm}) of the group $S(M) = Sp(2, 2)$ can be seen as being analogous to the Wigner ones $(s, \pm m)$ of the Dirac theory in Minkowski spacetime.

However, in general, the above equivalent spinor UIRs may not coincide since the expressions of their basis generators are strongly dependent on the arbitrary phase factors of the fundamental spinors whether these depend on \vec{p} . Thus if we change

$$U_{\vec{p}, \sigma} \rightarrow e^{i\chi^+(\vec{p})} U_{\vec{p}, \sigma}, \quad V_{\vec{p}, \sigma} \rightarrow e^{-i\chi^-(\vec{p})} V_{\vec{p}, \sigma}, \quad (132)$$

with $\chi^{\pm}(\vec{p}) \in \mathbb{R}$, performing simultaneously the associated transformations,

$$\alpha(\vec{p}) \rightarrow e^{-i\chi^+(\vec{p})} \alpha(\vec{p}), \quad \beta(\vec{p}) \rightarrow e^{-i\chi^-(\vec{p})} \beta(\vec{p}), \quad (133)$$

that preserves the form of ψ , we find that the operators \tilde{P}_i keep their forms while the other generators are changing, e. g. the Hamiltonian operators transform as, $\tilde{H}^{(\pm)} \rightarrow \tilde{H}^{(\pm)} + p^i \partial_{p^i} \chi^{\pm}(\vec{p})$. Obviously, these transformations are nothing other than unitary transformations among equivalent UIRs. Note that thanks to this mechanism one can fix suitable phases for determining desired forms of the basis generators keeping thus under control the flat and rest limits of these operators in the Dirac [16] or scalar [13, 25] field theory on M .

At the level of QFT, the operators $\{\mathbf{X}_{(AB)}\}$, given by Eq. (118) where we introduce the differential operators (120) -(123), generate a reducible operator valued CR which can be decomposed as the orthogonal sum of CRs - generated by $\{\mathbf{X}_{(AB)}^{(+)}\}$ and $\{\mathbf{X}_{(AB)}^{(-)}\}$ - that are equivalent between themselves and equivalent to the UIRs $(\frac{1}{2}, \nu_{\pm})$ of the $sp(2, 2)$ algebra. These one-particle operators are the principal conserved quantities of the Dirac theory corresponding to the de Sitter isometries via Noether theorem.

It is remarkable that in our formalism we have $\tilde{X}_{AB}^{(+)} = \tilde{X}_{AB}^{(-)}$ which means that the particle and antiparticle sectors bring similar contributions such that we can say that these quantities are *additive*, e. g., the energy of a many particle system is the sum of the individual energies of particles and antiparticles.

Other important conserved one-particle operators are the components

of the Pauli -Lubanski operator,

$$\mathbf{W}_A = \mathbf{W}_A^{(+)} + \mathbf{W}_A^{(-)} = \int d^3p \left[\alpha^\dagger(\vec{p}) \tilde{W}_A^{(+)} \alpha(\vec{p}) + \beta^\dagger(\vec{p}) \tilde{W}_A^{(-)} \beta(\vec{p}) \right], \quad (134)$$

as given by Eqs. (127)-(129). The Casimir operators of QFT have to be calculated according to Eqs. (125) and (126) but by using the one-particle operators $\mathbf{X}_{(AB)}$ and \mathbf{W}_A instead of $\tilde{X}_{(AB)}$ and \tilde{W}_A . We obtain the following one-particle contributions

$$\mathbf{C}_1 = \left(\mu^2 + \frac{3}{2} \right) \mathbf{N} + \dots, \quad \mathbf{C}_2 = \frac{3}{4} \left(\mu^2 + \frac{1}{4} \right) \mathbf{N} + \dots, \quad (135)$$

where $\mathbf{N} = \mathbf{N}^{(+)} + \mathbf{N}^{(-)}$ is the usual operator of the total number of particles and antiparticles.

Thus the additivity holds for the entire theory of the spacetime symmetries in contrast with the conserved charges of the internal symmetries that take different values for particles and antiparticles as, for example, the charge operator corresponding to the $U(1)_{em}$ gauge symmetry [18] that reads $\mathbf{Q} = q(\mathbf{N}^{(+)} - \mathbf{N}^{(-)})$.

6 Concluding remarks

The principal conclusion is that the QFT on the de Sitter background has similar features as in the flat case. Thus the covariant quantum fields transform according to CRs induced by the reps. of the group $\hat{G} = SL(2, \mathbb{C})$ that must be equivalent to orthogonal sums of UIRs of the group $S(M) = Sp(2, 2)$ whose specific invariants depend only on particle masses and spins. The example is the spinor CR of the Dirac theory that is induced by the linear rep. $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2})$ of the group \hat{G} but is equivalent to the orthogonal sum of two equivalent UIRs of the group $S(M)$ labelled by $(\frac{1}{2}, \nu_\pm)$. Thus at least in the case of the Dirac field we recover a similar conjuncture as in the Wigner theory of the induced reps. of the Poincaré group in special relativity. However, the principal difference is that the transformations of the Wigner UIRs can be written in closed forms while in our case this cannot be done because of the technical difficulties in solving the integrals (115) and (116). For this reason we were forced to restrict ourselves to study only the reps. of the corresponding algebras.

This is not an impediment since physically speaking we are interested to know the properties of the basis generators (in configurations or momentum rep.) since these give rise to the conserved observables (i. e. the

one-particle operators) of QFT, associated to the de Sitter isometries. It is remarkable that the particle and antiparticle sectors of these operators bring additive contributions since the particle and antiparticle operators transform alike under isometries just as in special relativity. Notice that this result was obtained by Nachtmann [13] for the scalar UIRs but this is less relevant as long as the generators of the scalar rep. depend only on m^2 . Now we see that the generators of the spinor rep. which have spin terms depending on m preserve this property such that we can conclude that all the one-particle operators corresponding to the de Sitter isometries are additive, regardless the spin.

The principal problem that remains unsolved here is how to build on the de Sitter manifolds a Wigner type theory able to define the structure of the covariant fields without using field equations. This means to solve first the problem of the UIR transformations in momentum rep. and then to look for a general definition of mass or even of a mass operator on M related to the Casimir operators of the UIRs of the $S(M)$ group. We note that despite of the well-known classical results [13, 26] there remains a discrepancy between the manners in which the masses of bosons and fermions depend on the de Sitter invariants [17]. We hope that the results presented here will offer one new tools in solving these delicate problems.

A Finite-dimensional representations of the $sl(2, \mathbb{C})$ algebra

The standard basis of the $sl(2, \mathbb{C})$ algebra is formed by the generators $\vec{J} = (J_1, J_2, J_3)$ and $\vec{K} = (K_1, K_2, K_3)$ that satisfy [27, 5]

$$[J_i, J_j] = i\varepsilon_{ijk}J_k, \quad [J_i, K_j] = i\varepsilon_{ijk}K_k, \quad [K_i, K_j] = -i\varepsilon_{ijk}J_k, \quad (136)$$

having the Casimir operators $c_1 = i\vec{J} \cdot \vec{K}$ and $c_2 = \vec{J}^2 - \vec{K}^2$. The linear combinations $A_i = \frac{1}{2}(J_i + iK_i)$ and $B_i = \frac{1}{2}(J_i - iK_i)$ form two independent $su(2)$ algebras satisfying

$$[A_i, A_j] = i\varepsilon_{ijk}A_k, \quad [B_i, B_j] = i\varepsilon_{ijk}B_k, \quad [A_i, B_j] = 0. \quad (137)$$

Consequently, any finite-dimensional irreducible rep. (IR) $\tau = (j_1, j_2)$ is carried by the space of the direct product $(j_1) \otimes (j_2)$ of the UIRs (j_1) and (j_2) of the $su(2)$ algebras (A_i) and respectively (B_i) . These IRs are labeled either by the $su(2)$ labels (j_1, j_2) or giving the values of the Casimir operators $c_1 = j_1(j_1 + 1) - j_2(j_2 + 1)$ and $c_2 = 2[j_1(j_1 + 1) + j_2(j_2 + 1)]$.

The fundamental reps. defining the $sl(2, \mathbb{C})$ algebra are either the IR $(\frac{1}{2}, 0)$ generated by $\{\frac{1}{2}\sigma_i, -\frac{i}{2}\sigma_i\}$ or the IR $(0, \frac{1}{2})$ whose generators are $\{\frac{1}{2}\sigma_i, \frac{i}{2}\sigma_i\}$. Their direct sum form the spinor IR $\rho_s = (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ of the Dirac theory. In applications it is convenient to consider ρ_s as the fundamental rep. since here invariant forms can be defined using the Dirac conjugation.

The spin basis of the IR τ can be constructed as the direct product,

$$|\tau, s\sigma\rangle = \sum_{\lambda_1 + \lambda_2 = \sigma} C_{j_1 \lambda_1, j_2 \lambda_2}^{s\sigma} |j_1, \lambda_1\rangle \otimes |j_2, \lambda_2\rangle, \quad (138)$$

of $su(2)$ canonical bases where the Clebsh-Gordan coefficients [5] give the spin content of the IR τ , i. e. $s = j_1 + j_2, j_1 + j_2 - 1, \dots, |j_1 - j_2|$. Note that for integer values of spin we can resort to the tensor bases constructed as direct products of the vector bases $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ of the vector IR ($j = 1$) (which satisfy $|1, \pm 1\rangle = \frac{1}{\sqrt{2}}(\vec{e}_1 \pm i\vec{e}_2)$ and $|1, 0\rangle = \vec{e}_3$).

Given the IR $\tau = (j_1, j_2)$ we say that its adjoint IR is $\dot{\tau} = (j_2, j_1)$ and observe that these have the same spin content while their generators are related as $\vec{J}^{(\dot{\tau})} = \vec{J}^{(\tau)}$ and $\vec{K}^{(\dot{\tau})} = -\vec{K}^{(\tau)}$. On the other hand, the operators \vec{A} and \vec{B} are Hermitian since we use UIRs of the $su(2)$ algebra. Consequently, we have $\vec{J}^+ = \vec{J}$ and $\vec{K}^+ = -\vec{K}$ for any finite-dimensional IR of the $sl(2, \mathbb{C})$ algebra, such that we can write

$$(\vec{J}^{(\tau)})^+ = \vec{J}^{(\tau)}, \quad (\vec{K}^{(\tau)})^+ = -\vec{K}^{(\tau)}. \quad (139)$$

Hereby we conclude that invariant forms can be constructed only when we use reducible reps. $\rho = \dots \tau_1 \oplus \tau_2 \dots \dot{\tau}_1 \oplus \dot{\tau}_2 \dots$ containing only pairs of adjoint reps.. Then the matrix $\gamma_{(\rho)}$ may be constructed with the matrix elements

$$\langle \tau_1, s_1 \sigma_1 | \gamma_{(\rho)} | \tau_2, s_2 \sigma_2 \rangle = \delta_{\tau_1 \dot{\tau}_2} \delta_{s_1 s_2} \delta_{\sigma_1 \sigma_2}. \quad (140)$$

Note that the canonical basis $\{\tau, j\lambda\}$ defines the *chiral* rep. while a new basis in which $\gamma_{(\rho)}$ becomes diagonal gives the so called standard rep.. This terminology comes from the Dirac theory where $\gamma_{(\rho_s)} = \gamma^0$ is the Dirac matrix that may have these reps. [24].

B Some properties of Hankel functions

According to the general properties of the Hankel functions [28], we deduce that those used here, $H_{\nu_{\pm}}^{(1,2)}(z)$, with $\nu_{\pm} = \frac{1}{2} \pm i\mu$ and $z \in \mathbb{R}$, are related among themselves through $[H_{\nu_{\pm}}^{(1,2)}(z)]^* = H_{\nu_{\mp}}^{(2,1)}(z)$ and satisfy the identities

$$e^{\pm \pi k} H_{\nu_{\mp}}^{(1)}(z) H_{\nu_{\pm}}^{(2)}(z) + e^{\mp \pi k} H_{\nu_{\pm}}^{(1)}(z) H_{\nu_{\mp}}^{(2)}(z) = \frac{4}{\pi z}. \quad (141)$$

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